

HESSIAN K3 SURFACES OF NON SYLVESTER TYPE

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ABSTRACT. We construct the moduli space of cubic surfaces which do not admit a Sylvester form as an arithmetic quotient, and determine the graded ring of modular forms of even weights.

1. HESSIAN K3 SURFACES OF NON SYLVESTER TYPE

1.1. It is classically known that the ring of $\mathrm{SL}_4(\mathbb{C})$ -invariants of quaternary cubic forms is

$$\mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}] \quad (\deg I_n = n)$$

where I_8, \dots, I_{40} are algebraically independent and $I_{100}^2 \in \mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}]$ ([H], [Sa]). Hence the moduli space of cubic surfaces \mathcal{M}_I is isomorphic to the weighted projective space

$$\mathrm{Proj} \mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}] = \mathbb{P}(1, 2, 3, 4, 5)_I.$$

A general cubic surface is written as a complete intersection

$$S_\lambda : X_0 + \dots + X_4 = 0, \quad \lambda_0 X_0^3 + \dots + \lambda_4 X_4^3 = 0$$

in \mathbb{P}^4 with $\lambda_0, \dots, \lambda_4 \neq 0$, which is called the Sylvester form. Let σ_i be the i -th elementary symmetric polynomial in $\lambda_0, \dots, \lambda_4$. They give invariants of S_λ , and we have

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_5^3\sigma_1, \quad I_{24} = \sigma_5^4\sigma_4, \quad I_{32} = \sigma_5^6\sigma_2, \quad I_{40} = \sigma_5^8.$$

This correspondence gives a birational map

$$\mathbb{P}(1, 2, 3, 4, 5)_\lambda \longrightarrow \mathbb{P}(1, 2, 3, 4, 5)_I$$

with the base locus $\sigma_5 = \sigma_4 = 0$. The Hessian of S_λ is given by

$$H_\lambda : X_0 + \dots + X_4 = 0, \quad \frac{1}{\lambda_0 X_0} + \dots + \frac{1}{\lambda_4 X_4} = 0.$$

The Picard lattice of the desingularization of a general H_λ is $U \oplus U(2) \oplus A_2(2)$ (see [DK]).

1.2. **Dardanelli - van Geemen's stratification.** The following facts on \mathcal{M} were proved in [DvG].

(I) The subvariety of \mathcal{M} parametrizing cubic surfaces which do not admit a Sylvester form is defined by $I_{40} = 0$. In general, such surfaces are given by

$$S_{ns1}(a) : X_1^3 + X_2^3 + X_3^3 - X_0^2(a_0 X_0 + 3a_1 X_1 + 3a_2 X_2 + 3a_3 X_3) = 0.$$

If we denote the i -th elementary symmetric polynomial in a_1^3, a_2^3, a_3^3 by ρ_i , then we have

$$[S_{ns1}(a)] = [-4\rho_1 + a_0^2 : \rho_2 : 2\rho_3 : \rho_1\rho_3 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_I.$$

The Hessian surface of $S_{ns1}(a)$ is given by

$$H_{ns1}(a) : X_0 X_1 X_2 X_3 (a_1 \frac{X_1}{X_0} + a_2 \frac{X_2}{X_0} + a_3 \frac{X_3}{X_0} + a_0 + a_1^2 \frac{X_0}{X_1} + a_2^2 \frac{X_0}{X_2} + a_3^2 \frac{X_0}{X_3}) = 0,$$

and the transcendental lattice of the desingularization of a general $H_{ns1}(a)$ is $T_{ns1} = U \oplus U(2) \oplus \langle -4 \rangle$.

In affine coordinates $[X_0 : X_1 : X_2 : X_3] = [1 : x/a_1 : y/a_2 : z/a_3]$, the equation of $H_{ns1}(a)$ is

$$xyz(x + y + z + a_0 + a_1^3 \frac{1}{x} + a_2^3 \frac{1}{y} + a_3^3 \frac{1}{z}) = 0.$$

(II) The subvariety of \mathcal{M} parametrizing cubic surfaces

$$S_{ns2}(b) : X_1^3 + X_2^3 + 2b_0 X_3^3 - 3X_3(b_1 X_1 X_3 + X_2 X_3 + X_0^2) = 0$$

is defined by $I_{24} = I_{40} = 0$, and we have

$$[S_{ns2}(b)] = [-8b_0 : 1 + b_1^3 : 0 : b_1^3 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_I.$$

The Hessian surface of $S_{ns2}(b)$ is given by

$$H_{ns2}(b) : X_1 X_2 X_3 (-2b_0 X_3 + b_1 X_1 + X_2) + X_3^3 (X_1 + b_1^2 X_2) - X_0^2 X_1 X_2 = 0$$

and the transcendental lattice of the desingularization of a general $H_{ns2}(b)$ is $T_{ns2} = U \oplus U(2)$.

(III) The subvariety of \mathcal{M} parametrizing “cyclic cubic surfaces”

$$S_{cyc}(a) : a_4 X_4^3 - a_3 (X_0 + X_1 + X_2)^3 + a_0 X_0^3 + a_1 X_1^3 + a_2 X_2^3 = 0$$

is defined by $I_{24} = I_{32} = I_{40} = 0$, and we have

$$[S_{cyc}(a)] \in [\mu_3^2 - 4\mu_2\mu_4 : \mu_4^3 : 0 : 0 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_I$$

where μ_i is the i -th symmetric polynomial of a_0, \dots, a_3 . The Hessian of $S_{cyc}(a)$ is reducible.

(IV) The strictly semi-stable surface $t^3 = xyz$ corresponds to the point $[8 : 1 : 0 : 0 : 0]$, and the Fermat cubic surface corresponds to the point $[1 : 0 : 0 : 0 : 0]$.

1.3. Batyrev’s mirror construction. Hessian surfaces $\{H_{ns1}\}$ are obtained also as toric hypersurfaces. Let Δ be the octahedron in \mathbb{R}^3 with vertices

$$(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1).$$

It is a simplicial reflexive polytope, and its dual polytope Δ^* is the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Considering faces of Δ as simplicial cones, we obtain a toric variety $X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The linear system of anti-canonical classes of $X(\Delta)$ (that is, K3 surfaces of degree $(2, 2, 2)$ in $(\mathbb{P}^1)^3$) is given by

$$\mathcal{F}(\Delta^*) = \{ \sum a_{ijk} x^i y^j z^k = 0 \mid (i, j, k) \in \Delta^* \cap \mathbb{Z}^3 \} \quad ((x, y, z) \in (\mathbb{C}^\times)^3 \subset (\mathbb{P}^1)^3).$$

Similarly, we have the dual family of K3 surfaces

$$\mathcal{F}(\Delta) = \{ c_1 x + c_2 y + c_3 z + c_4 + c_5 \frac{1}{x} + c_6 \frac{1}{y} + c_7 \frac{1}{z} = 0 \}$$

as hypersurfaces of $X(\Delta^*)$. It is obvious that this family is birationally equivalent to the family $\{H_{ns1}\}$. Note that the Picard lattice of a general member of $\mathcal{F}(\Delta^*)$ is

$$P = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \cong U(2) \oplus \langle -4 \rangle,$$

and we have $T_{ns1} = U \oplus P$. Hence $\mathcal{F}(\Delta)$ is the mirror partner of $\mathcal{F}(\Delta^*)$ (see [B], [C1], [D1] and [GN]). Note also that $\mathcal{F}(\Delta)$ is a subfamily of $\mathcal{F}(\Delta^*)$. In the following, we regard H_{ns1} as hypersurfaces in $(\mathbb{P}^1)^3$, and we replace coefficients a_0, a_1^3, a_2^3, a_3^3 of H_{ns1} by $1, u_1, u_2, u_3$:

$$H(u) : f_u = xyz(x + y + z + 1) + (u_1 yz + u_2 zx + u_3 xy) = 0 \quad (x, y, z) \in (\mathbb{P}^1)^3.$$

1.4. Remark. From the 1-parameter family

$$H_{PS}(u) : xyz(x + y + z + 1) + u(xy + yz + zx) = 0,$$

by the base change $u = (t + t^{-1})^{-2}$, we obtain the family

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + t + \frac{1}{t} = 0$$

studied by Peters and Stienstra in [PS]. They studied the Picard-Fuchs equation and modular forms. The transcendental lattice of a general member is $U \oplus \langle 12 \rangle$. This K3-fibration is considered as a (singular) Calabi-Yau hypersurface in $(\mathbb{P}^1)^4$ (see [V]).

1.5. Singularities. Let us assume $u_1 u_2 u_3 \neq 0$. Then $H(u) \cap (\mathbb{C}^\times)^3$ is smooth if and only if

$$\Delta_{sing}(u) = \prod (1 \pm 2\sqrt{u_1} \pm 2\sqrt{u_2} \pm 2\sqrt{u_3}) \neq 0.$$

Therefore we define the parameter space

$$\mathcal{U} = \{u = (u_1, u_2, u_3) \mid u_1 u_2 u_3 \Delta_{sing}(u) \neq 0\}.$$

For any $u \in \mathcal{U}$, we see that $H(u) \cap ((\mathbb{P}^1)^3 - (\mathbb{C}^\times)^3)$ is decomposed into twelve lines

$$L_{x00} = \mathbb{P}^1 \times \{0\} \times \{0\}, \quad L_{x0\infty} = \mathbb{P}^1 \times \{0\} \times \{\infty\}, \quad \dots, \quad L_{\infty\infty z} = \{\infty\} \times \{\infty\} \times \mathbb{P}^1.$$

They intersect at eight points

$$(0, 0, 0), \quad (0, 0, \infty), \quad (0, \infty, 0), \quad (0, \infty, \infty), \\ (\infty, 0, 0), \quad (\infty, 0, \infty), \quad (\infty, \infty, 0), \quad (\infty, \infty, \infty),$$

that are singular points of $H(u)$, and all of them are A_1 -singularities. Blowing up eight singular points of $H(u)$, we obtain a K3 surface $\tilde{H}(u)$. Let $N_u \subset H^2(\tilde{H}(u), \mathbb{Z})$ be a sublattice generated by twelve lines $L_{x00}, \dots, L_{\infty\infty z}$ and eight exceptional curves $E_{000}, \dots, E_{\infty\infty\infty}$ that are blown down to $(0, 0, 0), \dots, (\infty\infty\infty)$.

1.6. Proposition. (1) For a general $u \in \mathcal{U}$, the lattice N_u is the Picard lattice $\text{Pic}(\tilde{H}(u))$.

(2) We have three involutions

$$\epsilon_x : (x, y, z) \mapsto \left(\frac{u_1}{x}, y, z\right), \quad \epsilon_y : (x, y, z) \mapsto \left(x, \frac{u_2}{y}, z\right), \quad \epsilon_z : (x, y, z) \mapsto \left(x, y, \frac{u_3}{z}\right).$$

on $\tilde{H}(u)$, and the product $\epsilon = \epsilon_x \epsilon_y \epsilon_z$ is an Enriques involution.

(3) Let $N_u^* \subset N \otimes \mathbb{Q}$ be the dual lattice of N_u , and $q_N : N_u^*/N_u \rightarrow \mathbb{Q}/2\mathbb{Z}$ be the discriminant form $([N])$. Then we have $\epsilon = \epsilon_x = \epsilon_y = \epsilon_z$ as elements of the finite orthogonal group $O(q_N)$. Moreover, we have $O(q_N) = S_3 \times \langle \epsilon \rangle$, where S_3 is realized as symmetry of (x, y, z) .

Proof. (1) The self intersection numbers of L_{***} and E_{***} are -2 , and we have $E_{abc} \cdot L_{stu} = 1$ if two of three equalities $a = s$, $b = t$ or $c = u$ are hold. Other intersection numbers are zero. Using a computer, we can show that the rank of the intersection matrix of them is 17. In fact, we have equalities

$$\begin{aligned} E_{000} &= E_{00\infty} + E_{0\infty 0} + 3E_{0\infty\infty} - 3E_{\infty 00} - E_{\infty 0\infty} - E_{\infty\infty 0} + E_{\infty\infty\infty} \\ &\quad - 2L_{x00} + 2L_{x\infty\infty} + 2L_{0y0} - 2L_{\infty y0} + 2L_{0\infty z} - 2L_{\infty 0z}, \\ L_{\infty y\infty} &= 2E_{00\infty} + 2E_{0\infty\infty} - 2E_{\infty 00} - 2E_{\infty 0\infty} - L_{x00} - L_{x0\infty} + L_{x\infty\infty} \\ &\quad + L_{x\infty\infty} + L_{0y0} + L_{0y\infty} - L_{\infty y0} + 2L_{0\infty z} - 2L_{\infty 0z}, \\ L_{\infty\infty z} &= 2E_{00\infty} + 2E_{0\infty\infty} - 2E_{\infty 00} - 2E_{\infty 0\infty} - L_{x00} + L_{x0\infty} - L_{x\infty\infty} \\ &\quad + L_{x\infty\infty} + 2L_{0y0} - 2L_{\infty y0} + L_{0\infty z} - L_{\infty 0z} \end{aligned}$$

as elements of N_u . Therefore E_{000} , $L_{\infty y\infty}$ and $L_{\infty\infty z}$ are redundant. Since the determinat of the intersection matrix of other 17 curves is 16, we see that they span the orthogonal complement of $T_{ns1} = U \oplus U(2) \oplus \langle -4 \rangle$.

(2) As an involution of $(\mathbb{P}^1)^3$, fixed points of ϵ are $(\pm\sqrt{u_1}, \pm\sqrt{u_2}, \pm\sqrt{u_3})$. If $u \in \mathcal{U}$, then such points are not on $H(u)$.

(3) We have $N_u^*/N_u \cong T_{ns1}^*/T_{ns1} \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$, and it is generated by

$$\begin{aligned} \ell_1 &= \frac{1}{2}(L_{0y0} + L_{0y\infty} + L_{00z} + L_{0\infty z}), \\ \ell_2 &= \frac{1}{2}(L_{x00} + L_{x0\infty} + L_{00z} + L_{\infty 0z}), \\ m &= \frac{1}{4}(2E_{00\infty} + 2E_{0\infty\infty} + 2E_{\infty\infty 0} + 2E_{\infty\infty\infty} + 2L_{x00} \\ &\quad + 3L_{x0\infty} + 3L_{x\infty\infty} + 2L_{0y0} + L_{0y\infty} + L_{\infty y0} + 3L_{0\infty z} + L_{\infty 0z}). \end{aligned}$$

By machine computation, we see that

$$\epsilon_x(\ell_i) = \ell_i \ (i = 1, 2), \quad \epsilon_x(m) = -m,$$

and the same for ϵ_y and ϵ_z . The 2-torsion subgroup of N_u^*/N_u is generated by ℓ_1, ℓ_2 and $\ell_3 = 2m + \ell_1 + \ell_2$, and these are all of elements $x \in N_u^*/N_u$ of order 2 such that $q_N(x) = 0$. We have a split exact sequence

$$1 \longrightarrow \langle \epsilon \rangle \longrightarrow O(q_N) \longrightarrow \{\text{permutations of } \ell_1, \ell_2, \ell_3\} \longrightarrow 1$$

since permutations of (x, y, z) give permutations of ℓ_i 's. \square

2. THE PERIOD MAPPING AND MODULAR GROUPS

2.1. The period mapping. The period domain of the family $\{\tilde{H}(u) \mid u \in \mathcal{U}\}$ is the bounded symmetric domain

$$\mathbb{D}_{ns} = \{z \in \mathbb{P}^4 \mid {}^t z Q z = 0, {}^t z Q \bar{z} > 0\}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \oplus [-4].$$

of type IV defined by the lattice T_{ns1} . More explicitly, we have

$$[1 : z_2 : \cdots : z_5] \in \mathbb{D}_{ns} \Leftrightarrow \begin{cases} z_2 = -2(z_3 z_4 - z_5^2) \\ y_3 y_4 - y_5^2 > 0 \ (y_i = \text{Im} z_i) \end{cases}$$

and $\mathbb{D}_{ns} = \mathbb{D}_{ns}^+ \amalg \mathbb{D}_{ns}^-$ where $\mathbb{D}_{ns}^\pm = \{z \in \mathbb{D}_{ns} : \pm y_3 > 0\}$. Let us define the orthogonal group

$$O_{ns}^+ = \{g \in \text{GL}_5(\mathbb{Z}) \mid {}^t g Q g = Q, g(\mathbb{D}_{ns}^+) = \mathbb{D}_{ns}^+\}$$

on the lattice T_{ns1} , which acts on \mathbb{D}_{ns}^+ . We define also the discriminant form

$$q_{ns1} : T_{ns1}^*/T_{ns1} \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

and the orthogonal group $O(q_{ns1})$. Let $O_{ns}^+(2)_\epsilon$ be the kernel of the natural homomorphism

$$O_{ns}^+ \longrightarrow O(q_{ns1}) \cong O(q_N) \cong S_3 \times \langle \epsilon \rangle,$$

and $O_{ns}^+(2)$ be the kernel of the composition map

$$O_{ns}^+(2)_\epsilon \longrightarrow S_3 \times \langle \epsilon \rangle \longrightarrow S_3.$$

We have $-1 \in O_{ns}^+(2)$ and $-1 \notin O_{ns}^+(2)_\epsilon$. Since $[O_{ns}^+(2)_\epsilon : O_{ns}^+(2)] = 2$, we see that

$$\mathbb{D}_{ns}^+/O_{ns}^+(2)_\epsilon = \mathbb{D}_{ns}^+/O_{ns}^+(2).$$

Let $S_u \subset H_2(\tilde{H}(u), \mathbb{Z})$ be the sublattice generated by L_{***} 's and E_{***} 's, that is, the Poincare dual of $N_u \subset H^2(\tilde{H}(u), \mathbb{Z})$. Taking suitable 2-cycles $\gamma_1(u), \dots, \gamma_5(u) \in (S_u)^\perp \cong T_{ns1}$ that are uniquely determined up to O_{ns}^+ -action, we can define the period mapping

$$\text{Per} : \mathcal{U} \longrightarrow \mathbb{D}_{ns}^+, \quad u = (u_1, u_2, u_3) \mapsto \left[\int_{\gamma_1(u)} \omega_u : \cdots : \int_{\gamma_5(u)} \omega_u \right]$$

where $\omega_u \in H^{2,0}(\tilde{H}(u))$.

2.2. Proposition. The multi-valued map Per induces an injective S_3 -equivariant map $\mathcal{U} \longrightarrow \mathbb{D}_{ns}^+/O_{ns}^+(2)_\epsilon$ and the map $\mathcal{U}/S_3 \longrightarrow \mathbb{D}_{ns}^+/O_{ns}^+$ for S_3 -quotients.

Proof. Note that

(1) the monodromy action of $\pi_1(\mathcal{U}, u)$ on $S_u \subset H_2(\tilde{H}(u), \mathbb{Z})$ is trivial,

(2) we can lift $g \in O_{ns}^+$ to $\tilde{g} \in O(H_2(\tilde{H}(u), \mathbb{Z}))$ such that $\tilde{g}|_{S_u} = \text{id}$ iff $g \in O_{ns}^+(2)_\epsilon$.

From these facts together with Proposition 1.6, we see that the map is injective as the period map of N_u -polarized K3-surfaces (see [D1]). \square

2.3. Proposition. The period map Per is given by the developping map of the Lauricella's hypergeometric differential equation for $F_C(1, \frac{1}{2}; 1, 1, 1; -2u_1, -2u_2, -2u_3)$ (see [Y]).

Proof. Indeed, we obtain a period of $H(u)$ as follows.

$$\begin{aligned}
I(u_1, u_2, u_3) &= \iiint_{|x|=|y|=|z|=\varepsilon} \frac{dx \wedge dy \wedge dz}{f_u} \\
&= \iiint_{|x|=|y|=|z|=\varepsilon} \frac{1}{xyz(x+y+z+1)} \frac{dxdydz}{1 + \frac{u_1xy+u_2yz+u_3zx}{xyz(x+y+z+1)}} \\
&= \iiint_{|x|=|y|=|z|=\varepsilon} \sum_{n=0}^{\infty} \frac{(-u_1xy - u_2yz - u_3zx)^n}{(xyz(x+y+z+1))^{n+1}} dxdydz \quad (|u_i| \ll \varepsilon) \\
&= \iiint_{|x|=|y|=|z|=\varepsilon} \sum_{p,q,r=0}^{\infty} \frac{(p+q+r)!}{p!q!r!} \frac{x^{p+r}y^{p+q}z^{q+r}dxdydz}{(xyz(x+y+z+1))^{p+q+r+1}} (-u_1)^p(-u_2)^q(-u_3)^r \\
&= \sum_{p,q,r=0}^{\infty} \frac{(p+q+r)!}{p!q!r!} N(p, q, r) (-u_1)^p(-u_2)^q(-u_3)^r
\end{aligned}$$

where

$$\begin{aligned}
N(p, q, r) &= \iiint_{|x|=|y|=|z|=\varepsilon} \frac{dxdydz}{x^{q+1}y^{r+1}z^{p+1}(x+y+z+1)^{p+q+r+1}} \\
&= (2\pi i)^3 \frac{(2p+2q+2r)!}{(p+q+r)!p!q!r!}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
I(u_1, u_2, u_3) &= (2\pi i)^3 \sum_{p,q,r=0}^{\infty} \frac{(2p+2q+2r)!}{(p!q!r!)^2} (-u_1)^p(-u_2)^q(-u_3)^r \\
&= \sum_{p,q,r=0}^{\infty} \frac{(1)_{p+q+r}(\frac{1}{2})_{p+q+r}}{(1)_p(1)_p(1)_q(1)_q(1)_r(1)_r} (-2u_1)^p(-2u_2)^q(-2u_3)^r \\
&= F_C(1, \frac{1}{2}; 1, 1, 1; -2u_1, -2u_2, -2u_3).
\end{aligned}$$

□

2.4. Modular groups. The domain \mathbb{D}_{ns}^+ is isomorphic to the Siegel upper half space \mathfrak{S}_2 of degree 2 by the map

$$\Psi : \mathbb{D}_{ns}^+ \longrightarrow \mathfrak{S}_2 = \{\tau \in \mathrm{GL}_2(\mathbb{C}) \mid \mathrm{Im}\tau > 0\}, \quad [1 : z_2 : \cdots : z_5] \mapsto \begin{bmatrix} z_3 & z_5 \\ z_5 & z_4 \end{bmatrix}.$$

The symplectic group

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \{g \in \mathrm{GL}_{2g}(\mathbb{R}) \mid {}^t g J g = J\}, \quad J = \begin{bmatrix} 0 & -I_g \\ I_g & 0 \end{bmatrix}$$

acts on \mathfrak{S}_g by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$. Let Γ_g be the Siegel modular group $\mathrm{Sp}_{2g}(\mathbb{R}) \cap \mathrm{GL}_{2g}(\mathbb{Z})$.

We consider the congruence subgroup

$$\Gamma_0(2)_g = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \mid C \equiv 0 \pmod{2} \right\},$$

and the extension $\Gamma_0^*(2)_2$ of $\Gamma_0(2)_2$ by a normalizer $W = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -I_2 \\ 2I_2 & 0 \end{bmatrix}$.

2.5. Proposition. Then we have an isomorphism $O_{ns}^+/\{\pm 1\} \cong \Gamma_0^*(2)_2/\{\pm 1\}$ as automorphisms of $\mathbb{D}_{ns}^+ \cong \mathfrak{S}_2$.

Proof. This is an easy consequence of Theorem 3.1 in [Ko2], and we omit the proof. We give just explicit correspondences of generators:

(1) The map $g : \mathrm{GL}_2(\mathbb{Z}) \rightarrow O_{ns}^+$,

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto I_2 \oplus \begin{bmatrix} a_1^2 & a_2^2 & 2a_1a_2 \\ a_3^2 & a_4^2 & 2a_3a_4 \\ a_1a_3 & a_2a_4 & a_1a_4 + a_2a_3 \end{bmatrix}$$

is a homomorphism such that $\mathrm{Ker} g = \{\pm 1\}$ and $\Psi(g(A) \cdot z) = \begin{bmatrix} A & 0 \\ 0 & {}_tA^{-1} \end{bmatrix} \cdot \Psi(z)$.

(2) Let \mathcal{B}_2 be the additive group of integral symmetric matrices of degree 2. Then the map $h : \mathcal{B}_2 \rightarrow O_{ns}^+$,

$$\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2m_1m_2 + 2m_3^2 & 1 & -2m_2 & -2m_1 & 4m_3 \\ m_1 & 0 & 1 & 0 & 0 \\ m_2 & 0 & 0 & 1 & 0 \\ m_3 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is a homomorphism such that $\Psi(h(B) \cdot z) = \begin{bmatrix} I_2 & B \\ 0 & I_2 \end{bmatrix} \cdot \Psi(z)$.

(3) For $w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus [-1] \in O_{ns}^+$, we have $\Psi(w \cdot z) = -\frac{1}{2}\Psi(z)^{-1} = W \cdot \Psi(z)$. \square

2.6. Proposition. (1) If $x, y \in \mathbb{Z}^5$ satisfy ${}^txQx = {}^tyQy = 0$ and ${}^txQy = 1$, then there exists a transformation $\gamma \in O_{ns}^+$ such that $\gamma \cdot x = e_1$ and $\gamma \cdot y = e_2$, where e_i is the i -th unit vector.

(2) For any primitive sublattice $M \cong U \oplus \langle 12 \rangle$ of T_{ns1} , there exists $\gamma \in O_{ns}^+$ such that $\gamma(M)$ is either

$$M_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(e_3 + 3e_4) \quad \text{or} \quad M_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(2e_3 + 2e_4 + e_5).$$

For any primitive sublattice $M' \cong U \oplus U(2)$ of T_{ns1} , there exists $\gamma' \in O_{ns}^+$ such that

$$\gamma'(M') = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.$$

(3) We have the following table for periods of special subfamilies:

	$\mathbb{P}(1 : 2 : 3 : 4 : 5)_I$	lattice	$\mathbb{D}_{ns}^+/O_{ns}^+$	$\mathfrak{S}_2/\Gamma_0^*(2)_2$
$H_{ns2}(b)$	$[-8b_0 : 1 + b_1^3 : 0 : b_1^3 : 0]$	$U \oplus U(2)$	$[1 : z_2 : z_3 : z_4 : 0]$	$\begin{bmatrix} z_3 & 0 \\ 0 & z_4 \end{bmatrix}$
$H_{PS}(u)$	$[-12u + 1 : 3u^2 : 2u^3 : 3u^4 : 0]$	$U \oplus \langle 12 \rangle$	$[1 : z_2 : 2z_5 : 2z_5 : z_5]$	$\begin{bmatrix} 2z_5 & z_5 \\ z_5 & 2z_5 \end{bmatrix}$

Proof. (1) This is shown by the same argument with Proposition 3.2 in [Ko2].

(2) By (1), there exists $\gamma \in O_{ns}^+$ and $x, y, z \in \mathbb{Z}$ such that

$$\gamma(M) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(xe_3 + ye_4 + ze_5), \quad xy - z^2 = 3.$$

Now the assertion for M follows from the facts:

(i) the integer solutions of the system of equations

$$xy = z^2 + 3, \quad |x| > |z|, \quad |y| > |z|$$

are $(2, 2, \pm 1)$ or $(-2, -2, \pm 1)$,

(ii) if $|x| < |z|$ or $|y| < |z|$, then multiplying

$$I_2 \oplus \begin{bmatrix} 1 & 1 & \pm 2 \\ 0 & 1 & 0 \\ 0 & \pm 1 & 1 \end{bmatrix}, \quad I_2 \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \pm 2 \\ \pm 1 & 0 & 1 \end{bmatrix} \in O_{ns}^+,$$

we can decrease the value of $|z|$.

The assertion for M' is easily shown by the same way.

(3) By (2), periods of $H_{ns2}(b)$ belong to the divisor $\{z_5 = 0\}$ in \mathbb{D}_{ns}^+ . Because surfaces $H_{PS}(u)$ don't belong to the family $\{H_{ns2}(b)\}$, their periods don't belong to $\mathbb{P}(M_1 \otimes \mathbb{C}) \subset \{z_5 = 0\}$. Hence periods of $H_{PS}(u)$ belong to $\mathbb{P}(M_2 \otimes \mathbb{C})$. \square

3. GRADED RING OF THETA CONSTANTS

3.1. Let Γ' be a subgroup of $\mathrm{Sp}_4(\mathbb{R})$. A holomorphic function $f(\tau)$ on \mathfrak{S}_2 is a modular form of weight k with respect to Γ' if it holds

$$f((A\tau + B)(C\tau + D)^{-1}) = \det(C\tau + D)^k f(\tau)$$

for any $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma'$. Let $M_k(\Gamma')$ be the vector space of such functions, and $A(\Gamma')_{\text{even}}$ be the graded ring $\oplus_{k=0}^{\infty} M_{2k}(\Gamma')$. The generators of the graded ring $A(\Gamma_0(2)_2)_{\text{even}}$ are given by theta constants

$$\theta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}^2} \exp[\pi i^t(n+a)\tau(n+a) + 2\pi i^t(n+a)b], \quad \tau \in \mathfrak{S}.$$

For simplicity, we denote $\theta_{a,b}$ by θ_{xyzw} if $a = {}^t(x/2, y/2)$ and $b = {}^t(z/2, w/2)$.

3.2. **Theorem**(Ibukiyama, [Ib]). Let us define modular forms

$$\begin{aligned} \vartheta &= (\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4)/4, & \phi_1 &= (\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011})^2, & \phi_2 &= (\theta_{0100}^4 - \theta_{0110}^4)^2/16384, \\ \chi &= (\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2/4096 \end{aligned}$$

of weight 2, 4, 4 and 6. Then the graded ring $A(\Gamma_0(2)_2)_{\text{even}}$ is a free algebra $\mathbb{C}[\vartheta, \phi_1, \phi_2, \chi]$, and

$$\mathrm{Proj} A(\Gamma_0(2)_2)_{\text{even}} \cong \mathbb{P}(2, 4, 4, 6).$$

3.3. **Lemma.** The zero divisor of the function $\chi(\tau)$ is $\Gamma_0(2)_2$ -orbit of

$$\mathbb{H} \times \mathbb{H} = \left\{ \tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \in \mathfrak{S} \mid \tau_2 = 0 \right\}$$

with multiplicity 1, and $\chi(\tau)$ is the unique non-trivial function in $M_6(\Gamma_0(2)_2)$ vanishing there.

Proof. The first assertion is proved by exactly the same way as in [Kl], p.116 - p.118. By the equality of theta constants of one variable $\theta_{00}^4 = \theta_{01}^4 + \theta_{10}^4$, we see that

$$\begin{aligned} \vartheta(\tau) &= (\theta_{00}^4(\tau_1) + \theta_{01}^4(\tau_1))(\theta_{00}^4(\tau_3) + \theta_{01}^4(\tau_3))/4, \\ \phi_1 &= \theta_{00}^4(\tau_1)\theta_{01}^4(\tau_1)\theta_{00}^4(\tau_3)\theta_{01}^4(\tau_3), \\ \phi_2 &= (\theta_{00}^4(\tau_1) - \theta_{01}^4(\tau_1))^2(\theta_{00}^4(\tau_3) - \theta_{01}^4(\tau_3))^2/16384 \end{aligned}$$

for $\tau \in \mathbb{H} \times \mathbb{H}$. Therefore $\vartheta^3, \vartheta\phi_1, \vartheta\phi_2$ are linearly independent on $\mathbb{H} \times \mathbb{H}$. \square

3.4. **Proposition.** The involution $W = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -I_2 \\ 2I_2 & 0 \end{bmatrix}$ acts on $A(\Gamma_0(2)_2)_{\text{even}}$ as follows

$$\begin{aligned} \vartheta(W \cdot \tau) &= (2 \det \tau)^2 \vartheta(\tau), & \phi_1(W \cdot \tau) &= 1024(2 \det \tau)^4 \phi_2(\tau) \\ \phi_2(W \cdot \tau) &= (2 \det \tau)^4 \phi_1(\tau)/1024, & \chi(W \cdot \tau) &= (2 \det \tau)^6 \chi(\tau). \end{aligned}$$

Therefore we have

$$A(\Gamma_0^*(2)_2)_{\text{even}} = \mathbb{C}[\vartheta, \phi, \chi, \psi], \quad \mathrm{Proj} A(\Gamma_0^*(2)_2)_{\text{even}} \cong \mathbb{P}(2 : 4 : 6 : 8)$$

where $\phi = \phi_1 + 1024\phi_2$ and $\psi = \phi_1\phi_2$.

Proof. By the following formula ([Ig1], p.408)

$$\begin{aligned}
\theta_{0000}^2(\tau/2) &= \theta_{0000}^2(\tau) + \theta_{1000}^2(\tau) + \theta_{0100}^2(\tau) + \theta_{1100}^2(\tau) \\
\theta_{0001}^2(\tau/2) &= \theta_{0000}^2(\tau) + \theta_{1000}^2(\tau) - \theta_{0100}^2(\tau) - \theta_{1100}^2(\tau) \\
\theta_{0010}^2(\tau/2) &= \theta_{0000}^2(\tau) - \theta_{1000}^2(\tau) + \theta_{0100}^2(\tau) - \theta_{1100}^2(\tau) \\
\theta_{0011}^2(\tau/2) &= \theta_{0000}^2(\tau) - \theta_{1000}^2(\tau) - \theta_{0100}^2(\tau) + \theta_{1100}^2(\tau) \\
\theta_{0100}^2(\tau/2) &= 2(\theta_{0000}\theta_{0100} + \theta_{1000}\theta_{1100})(\tau) \\
\theta_{0110}^2(\tau/2) &= 2(\theta_{0000}\theta_{0100} - \theta_{1000}\theta_{1100})(\tau) \\
\theta_{1000}^2(\tau/2) &= 2(\theta_{0000}\theta_{1000} + \theta_{0100}\theta_{1100})(\tau) \\
\theta_{1001}^2(\tau/2) &= 2(\theta_{0000}\theta_{1000} - \theta_{0100}\theta_{1100})(\tau) \\
\theta_{1100}^2(\tau/2) &= 2(\theta_{0000}\theta_{1100} + \theta_{0100}\theta_{1000})(\tau) \\
\theta_{1111}^2(\tau/2) &= 2(\theta_{0000}\theta_{1100} - \theta_{0100}\theta_{1000})(\tau)
\end{aligned}$$

we see that

$$\vartheta(W \cdot \tau) = (\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4)(-\tau^{-1}/2)/4 = (\theta_{0000}^4 + \theta_{1000}^4 + \theta_{0100}^4 + \theta_{1100}^4)(-\tau^{-1}).$$

Applying the inversion formula, we obtain

$$\vartheta(W \cdot \tau) = 4(\det \tau)^2 \vartheta(\tau).$$

By the same way, we can show that

$$\phi_2(-\tau^{-1}/2) = (\det \tau)^4 \phi_1(\tau)/64,$$

and replacing τ by $-\tau^{-1}/2$, we see that

$$\phi_1(-\tau^{-1}/2) = 16384(\det \tau)^4 \phi_2(\tau).$$

For the modular form $\chi(\tau)$, we have

$$\frac{64}{(\det \tau)^6} \chi(-\tau^{-1}/2) = (\theta_{0000}^2\theta_{0001}^2 - \theta_{0010}^2\theta_{0011}^2)(\theta_{0000}^2\theta_{0010}^2 - \theta_{0001}^2\theta_{0011}^2)(\theta_{0000}^2\theta_{0011}^2 - \theta_{0001}^2\theta_{0010}^2)(\tau).$$

Since the right hand side vanishes on $\mathbb{H} \times \mathbb{H}$, it coincides with $c\chi(\tau)$ for some constant c . Comparing Fourier coefficients, we see that $c = 1$. \square

4. BOUNDARY

4.1. Let us study the extension of the period map $Per : \mathcal{U} \rightarrow \mathbb{D}_{ns}^+$ to the locus $\{u_3 = 0\}$. Note that we have

$$F_C(1, \frac{1}{2}; 1, 1, 1; -2u_1, -2u_2, 0) = F_4(1, \frac{1}{2}; 1, 1; -2u_1, -2u_2)$$

where F_4 is Appell's hypergeometric series, and we have

$$F_4(1, \frac{1}{2}; 1, 1; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}) = (1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}} {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; xy)$$

(see [E]). It is known that Gauss's hypergeometric series ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; t)$ has an elliptic integral representation. In deed, the same computation in Proposition 2.3 shows that $F_4(1, \frac{1}{2}; 1, 1; -2u_1, -2u_2)$ is a period integral of a curve

$$C(u) : xy(x+y+1) + u_1y + u_2x = 0$$

of degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The relation between this family and Appell's F_4 was already studied by Stienstra in [St]. Here we study the relation between the invariants of $C(u)$ and the degeneration of the map

$$DvG : \mathcal{U} \rightarrow \mathbb{P}(1, 2, 3, 4), \quad (u_1, u_2, u_3) \mapsto [-4s_1 + 1 : s_2 : 2s_3 : s_1s_3]$$

defined by invariants of cubic surfaces, where s_i is the i -th symmetric polynomial of u_1, u_2, u_3 . For this map, we have

$$\lim_{u_3 \rightarrow 0} [-4s_1 + 1 : s_2 : 2s_3 : s_1s_3] = [1 - 4u_1 - 4u_2 : u_1u_2 : 0 : 0].$$

On the other hand, the curve $C(u)$ is birationally equivalent to an elliptic curve

$$E(u) : Y^2 = f_u(X) = X^4 + X^3 + (-u_2 + \frac{u_1}{2} + \frac{1}{4})X^2 + \frac{u_1}{4}X + \frac{u_1^2}{16}$$

by the transformation

$$(x, y) = (2X, \frac{4Y - 4X^2 - 2X - u_1}{4X}).$$

4.2. Lemma. The classical invariants of the quartic equation $f_u(X) = 0$ are

$$\begin{aligned} g_2(u) &= \frac{1}{192}((1 - 4u_1 - 4u_2)^2 - 48u_1u_2), \\ g_3(u) &= -\frac{1}{13824}(1 - 4u_1 - 4u_2)((1 - 4u_1 - 4u_2)^2 - 72u_1u_2), \\ \Delta_E(u) &= g_2(u)^3 - 27g_3(u)^2 = \frac{1}{4096}u_1^2u_2^2((1 - 4u_1 - 4u_2)^2 - 64u_1u_2). \end{aligned}$$

Therefore $[1 - 4u_1 - 4u_2 : u_1u_2] \in \mathbb{P}(1, 2)$ corresponds to a singular $E(u)$ iff

$$[1 - 4u_1 - 4u_2 : u_1u_2] = [1 : 0] \text{ or } [8 : 1].$$

Moreover we have $\Delta_{sing}(u_1, u_2, 0) = (4096\Delta_E(u_1, u_2)/u_1^2u_2^2)^2$.

Proof. This is obtained from the definition

$$g_2 = ae - 4bd + 3c^2, \quad g_3 = \det \begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}$$

for $aX^4 + 4bX^3 + 6cX^2 + 4dX + e = 0$. □

4.3. Now we can define a degenerated period map

$$Per_{12} : \mathcal{U}_{12} = \{(u_1, u_2) \in \mathbb{C}^2 \mid \Delta_E(u_1, u_2) \neq 0\} \longrightarrow \mathbb{H},$$

and construct the inverse map

$$\mathbb{H} \longrightarrow [1 - 4u_1 - 4u_2 : u_1u_2] \in \mathbb{P}(1, 2)$$

by the Siegel Φ -operator $\Phi(f)(\tau_1) = \lim_{t \rightarrow \infty} f\left(\begin{bmatrix} \tau_1 & 0 \\ 0 & it \end{bmatrix}\right)$. Let us define modular forms

$$\begin{aligned} h_1 &= \Phi(8\vartheta) = 4(\theta_{00}^4 + \theta_{01}^4), \\ h_2 &= \Phi(\vartheta^2 - \phi) = \frac{1}{4}(\theta_{00}^4 + \theta_{01}^4)^2 - \theta_{00}^4\theta_{01}^4 = \frac{1}{4}(\theta_{00}^4 - \theta_{01}^4)^2 \end{aligned}$$

of weight 2 and 4 with respect to $\Gamma_0(2)_1$.

4.4. **Lemma.** Modular forms h_1 and h_2 satisfy same relations for $1 - 4u_1 - 4u_2$ and u_1u_2 in Lemma 4.2. In deed, we have

$$\begin{aligned} h_1(\tau)^2 - 48h_2(\tau) &= 64E_4(2\tau) = 64(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}), \\ h_1(\tau)(h_1(\tau)^2 - 72h_2(\tau)) &= -512E_6(2\tau) = -512(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}), \\ h_2(\tau)^2(h_1(\tau)^2 - 64h_2(\tau)) &= 2^{18}\eta(2\tau) = 2^{18}q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^{24}, \\ h_2(\tau)/(h_1(\tau)^2 - 64h_2(\tau)) &= \eta(2\tau)/\eta(\tau) \end{aligned}$$

where $q = \exp(2\pi i\tau)$, and

$$\lim_{t \rightarrow \infty} [h_1(it) : h_2(it)] = [1 : 0], \quad \lim_{t \rightarrow \infty} [h_1(-1/2it) : h_2(-1/2it)] = [8 : 1] \in \mathbb{P}(2, 4).$$

Since $\eta(2\tau)/\eta(\tau)$ is the Hauptmodul for $\Gamma_0(2)_1$, we see that the map

$$\mathbb{H}/\Gamma_0(2)_1 \cup \{0, \infty\} \rightarrow \mathbb{P}(1, 2), \quad \tau \mapsto [h_1(\tau) : h_2(\tau)] = [1 - 4u_1 - 4u_2 : u_1u_2]$$

is an isomorphism.

Proof. By the formula

$$\theta_{00}^2(2\tau) = \frac{1}{2}(\theta_{00}^2(\tau) + \theta_{01}^2(\tau)), \quad \theta_{01}^2(2\tau) = \theta_{00}(\tau)\theta_{01}(\tau), \quad \theta_{10}^2(2\tau) = \frac{1}{2}(\theta_{00}^2(\tau) - \theta_{01}^2(\tau)),$$

we have

$$\begin{aligned} E_4(2\tau) &= [\theta_{00}^8 - \theta_{00}^4\theta_{01}^4 + \theta_{01}^8](2\tau) \\ &= \frac{1}{16}[(\theta_{00}^2 + \theta_{01}^2)^4 - 4(\theta_{00}^2 + \theta_{01}^2)^2\theta_{00}^2\theta_{01}^2 + 16\theta_{00}^4\theta_{01}^4](\tau) \\ &= \frac{1}{64}[h_1^2 - 48h_2](\tau) \end{aligned}$$

and

$$\begin{aligned} E_6(2\tau) &= -\frac{1}{2}[(\theta_{00}^4 + \theta_{01}^4)(2\theta_{00}^4 - \theta_{01}^4)(\theta_{00}^4 - 2\theta_{01}^4)](2\tau) \\ &= -\frac{1}{64}[(\theta_{00}^4 + 6\theta_{00}^2\theta_{01}^2 + \theta_{01}^4)(\theta_{00}^4 + \theta_{01}^4)(\theta_{00}^4 - 6\theta_{00}^2\theta_{01}^2 + \theta_{01}^4)](\tau) \\ &= -\frac{1}{512}[h_1(h_1^2 - 72h_2)](\tau). \end{aligned}$$

Other assertions are shown by similar calculation.

4.5. **Theorem.** Let us define an embedding $\Theta : \mathfrak{S}_2/\Gamma_0^*(2)_2 \rightarrow \mathbb{P}(1, 2, 3, 4)$ by

$$\tau \mapsto [8\vartheta : \vartheta^2 - \phi : 1024\chi : 1024(\psi - \vartheta\chi)]$$

Then we have the commutative diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{Per} & \mathbb{D}_{ns}^+ \\ DvG \downarrow & & \downarrow \\ \mathbb{P}(1, 2, 3, 4) & \xleftarrow{\Theta} & \mathfrak{S}/\Gamma_0^*(2)_2 \cong \mathbb{D}_{ns}^+/\mathcal{O}_{ns}^+ \end{array}$$

and Θ induces an isomorphism $\mathfrak{S}_2/\Gamma_0^*(2)_2 \cup \mathbb{H}/\Gamma_0(2)_1 \cup \{0, \infty\} \cong \mathbb{P}(1, 2, 3, 4)$.

Proof. In deed, the map Θ is the unique map

$$\mathfrak{S}/\Gamma_0^*(2)_2 \rightarrow \mathbb{P}(1, 2, 3, 4), \quad \tau \mapsto [F_2(\tau) : F_4(\tau) : F_6(\tau) : F_8(\tau)] \quad (F_k \in M_k(\Gamma_0^*(2)_2))$$

such that

- (i) F_6 vanishes on $\mathbb{H} \times \mathbb{H}$,
- (ii)

$$\lim_{t \rightarrow \infty} [F_2 : F_4 : F_6 : F_8] \left(\begin{bmatrix} \tau & 0 \\ 0 & i\infty \end{bmatrix} \right) = [h_1(\tau) : h_2(\tau) : 0 : 0],$$

- (iii)

$$\begin{aligned} [F_2 : F_4 : F_6 : F_8] \left(\begin{bmatrix} 2\tau & \tau \\ \tau & 2\tau \end{bmatrix} \right) &= [-4s_1 + 1 : s_2 : 2s_3 : s_1 s_3]_{u_1=u_2=u_3=u} \\ &= [-12u + 1 : 3u^2 : 2u^3 : 3u^4], \end{aligned}$$

that is, $F_4^2 = 3F_8$ and $9F_6^2 = 4F_4F_8$.

For (i), we see that $F_6 = c\chi$ by Proposition 2.6 and Lemma 3.3. For (ii), note that

$$\begin{aligned} \lim_{t \rightarrow \infty} (c_1\vartheta^4 + c_2\vartheta^2\phi + c_3\phi^2 + c_4\vartheta\chi + c_5\psi) \left(\begin{bmatrix} \tau_1 & 0 \\ 0 & it \end{bmatrix} \right) &= 0 \\ \Leftrightarrow c_1(\theta_{00}^4 + \theta_{01}^4)^4/16 + c_2(\theta_{00}^4 + \theta_{01}^4)^2\theta_{00}^4\theta_{01}^4/4 + c_3(\theta_{00}^4\theta_{01}^4)^2 &= 0 \\ \Leftrightarrow c_1 = c_2 = c_3 = 0 \end{aligned}$$

Therefore we have

$$[F_2 : F_4 : F_6 : F_8] = [8\vartheta : \vartheta^2 - \phi : c\chi : c_4\vartheta\chi + c_5\psi].$$

Now the condition (iii) implies

$$(\vartheta^2 - \phi)^2(\tau) = 3(c_4\vartheta\chi + c_5\psi)(\tau), \quad 9(c\chi)^2(\tau) = [4(\vartheta^2 - \phi)(c_4\vartheta\chi + c_5\psi)](\tau)$$

for $\tau = \begin{bmatrix} 2\tau_1 & \tau_1 \\ \tau_1 & 2\tau_1 \end{bmatrix}$. Comparing Fourier coefficients

$$\begin{aligned} \vartheta(\tau) &= 1 + 72q^8 + 192q^{12} + 504q^{16} + 576q^{20} + 2280q^{24} + \cdots, \\ \phi(\tau) &= q^8 - 4q^{12} - 2q^{16} + 20q^{20} + 5q^{24} + \cdots, \\ \chi(\tau) &= q^{12} - 6q^{16} + 3q^{20} + 40q^{24} + \cdots, \\ \psi(\tau) &= q^{12} + 6q^{16} - 21q^{20} - 56q^{24} + \cdots, \end{aligned}$$

we obtain $c = 1024$, $c_4 = -1024$ and $c_5 = 1024$.

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